

# A note on the ramification of torsion points lying on curves of genus at least two

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## Abstract

Let  $C$  be a curve of genus  $g \geq 2$  defined over the fraction field  $K$  of a complete discrete valuation ring  $R$  with algebraically closed residue field. Suppose that  $\text{char}(K) = 0$  and that the characteristic of the residue field is not 2. Suppose that the Jacobian  $\text{Jac}(C)$  has semi-stable reduction over  $R$ . Embed  $C$  in  $\text{Jac}(C)$  using a  $K$ -rational point. We show that the coordinates of the torsion points lying on  $C$  lie in the unique moderately ramified quadratic extension of the field generated over  $K$  by the coordinates of the  $p$ -torsion points on  $\text{Jac}(C)$ .

## 1 Introduction

Let  $R$  be a complete discrete valuation ring. Suppose that the residue field  $k$  of  $R$  is algebraically closed and of characteristic  $p \geq 0$ . Suppose that  $p \neq 2$ . Let  $K$  be the fraction field of  $R$  and suppose that  $\text{char}(K) = 0$ . Let  $C$  be a curve of genus  $g \geq 2$  defined over  $K$ . Let  $j : C \rightarrow \text{Jac}(C)$  be the closed immersion of  $C$  into its Jacobian defined by a  $K$ -rational point. Let  $A := \text{Jac}(C)$ . Let  $\mathcal{A}$  be the Néron model of  $A$  over  $R$ . Suppose that the connected component of the special fiber  $\mathcal{A}_k$  of  $\mathcal{A}$  is a semi-abelian variety (in other words,  $\mathcal{A}$  has semi-stable reduction).

Let  $L := K(A[p](\bar{K}))$  be the extension of  $K$  generated by the coordinates of the  $p$ -torsion points of  $A(\bar{K})$ . In particular,  $L = K$  if  $p = 0$ . Let  $L'$  be the unique moderately ramified quadratic extension of  $L$ .

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Finally, let  $K_1 \subseteq \bar{K}$  be the field generated over  $K$  by the coordinates of the elements of  $\text{Tor}(A(\bar{K})) \cap C(\bar{K})$ . Here  $\text{Tor}(A(\bar{K}))$  is the subgroup of  $A(\bar{K})$  consisting of elements of finite order.

The aim of this note is to prove the following statement :

**Theorem 1.1. (a)**  $K_1 \subseteq L'$

**(b)** *if  $C$  is not hyperelliptic then  $K_1 \subseteq L$ .*

Theorem 1.1 should be understood as a complement to some results of Tamagawa (see [10]), Baker-Ribet (see [2]) and Coleman (see [5]).

For instance, with the present notation, suppose that  $p > 0$ , that  $R$  is the maximal unramified extension of  $\mathbb{Q}_p$  and that the abelian part of the connected component of  $\mathcal{A}_k$  is an ordinary abelian variety. Tamagawa then proves that  $K_1$  is contained in the extension of  $K$  generated by the  $p$ -th roots of unity (see [10] or [2, Th. 4.1]). Another example is the following result of Coleman : if  $p > \max(2g, 5)$ ,  $R$  is the maximal unramified extension of  $\mathbb{Q}_p$  and  $\mathcal{A}_k$  is an abelian variety, then  $K_1 \subseteq K$  (see [5, Conj. B]).

All these results restrict the size of  $K_1$  under various hypotheses on the special fiber  $\mathcal{A}_k$  and on the order of absolute ramification of  $K$ . The interest of Theorem 1.1 is that it provides a limit for the size of  $K_1$  under the mild hypothesis of semi-stability of  $\mathcal{A}$  only and with no assumption on the absolute ramification of  $K$ . The hypothesis of semi-stability is not very restrictive, because it will automatically be satisfied, if the  $l$ -torsion points of the Jacobian variety are  $K$ -rational, for  $l$  a prime number such that  $l > 2$  and  $l \neq p$  (Raynaud's criterion, see [6, IX]).

**Remark.** The avoidance of the prime  $p = 2$  is critical. It appears in both Lemma 2.4 and Lemma 2.6 and this is exploited at the end of the proof of Theorem 1.1. It would be interesting to extend this proof to the case  $p = 2$ .

**Notations.** If  $l$  is a prime number and  $G$  is an abelian group, we write  $\text{Tor}^l(G)$  for the set of elements of  $\text{Tor}(G)$  whose order is prime to  $l$  and  $\text{Tor}_l(G)$  for the set of elements of  $\text{Tor}(G)$  whose order is a power of  $l$ . The expression  $\text{Tor}^0(G)$  will stand for  $\text{Tor}(G)$ . We shall denote by  $+$  the group law on  $A(\bar{L})$ . We shall write divisors on  $C_{\bar{L}}$  in the form

$$n_1 P_1 \oplus n_2 P_2 \oplus \cdots \oplus n_r P_r$$

where  $n_i \in \mathbb{Z}$ . The symbol  $\sim$  will be used to denote linear equivalence of divisors.

## 2 Proof of Theorem 1.1

Let  $L^t$  be the maximal moderately ramified extension of  $L$ . Let  $I := \text{Gal}(\bar{L}|L)$ ,  $I^w := \text{Gal}(\bar{L}|L^t)$  and  $I^t := \text{Gal}(L^t|L)$ . Recall that  $I^w = 0$  if  $\text{char}(k) = 0$  and that there is a non-canonical isomorphism  $I^t \simeq \bigoplus_{l \neq p, l \text{ prime}} \mathbb{Z}_l$  (see [9, chap. IV]). Furthermore, the group  $I^w$  is a pro- $p$ -group if  $p > 0$ .

We shall need the following five results.

**Theorem 2.1** (monodromy theorem). *For any  $x \in \text{Tor}^p(A(\bar{L}))$  and any  $\sigma \in I$ , the equation  $\sigma^2(x) - 2\sigma(x) + x = 0$  is satisfied.*

**Proof.** See [6, IX, 5.12.2]  $\square$

**Lemma 2.2.** *The action of  $I^w$  on  $\text{Tor}^p(A(\bar{L}))$  is trivial.*

**Proof.** This is a direct consequence of Theorem 2.1. See for instance [2, Appendix, Lemma A.1].  $\square$

**Lemma 2.3.** *The action of  $I^t$  on  $\text{Tor}_p(A(L^t))$  is trivial.*

**Proof.** We may restrict ourselves to the case where  $p > 0$ . Let  $T \subseteq \text{Tor}_p(A(L^t))$  be a finite  $I^t$ -invariant subgroup. We have to show that the action of  $I^t$  on  $T$  is trivial. The action of  $I^t$  on  $T$  preserves the order of elements, hence  $T$  is an inner direct sum of  $G^t$ -invariants subgroups of the form  $(\mathbb{Z}/p^r)^s$ . Hence we might suppose without loss of generality that  $T \simeq (\mathbb{Z}/p^r)^s$  for some  $r, s \leq 1$ . Let  $T_p$  be the subgroup of  $p$ -torsion elements of  $T$ . The fact that the  $p$ -torsion points in  $A(\bar{L})$  are  $L$ -rational implies that the action of  $G^t$  on  $T_p$  is trivial. Hence the image of  $G^t$  in  $\text{Aut}(T)$  lies in the kernel of the natural group map

$$\text{Aut}(T) \rightarrow \text{Aut}(T_p)$$

Under the above isomorphism  $T \simeq (\mathbb{Z}/p^r)^s$ , this corresponds to  $s \times s$ -matrices of the form  $\text{Id} + pM$ , where  $M$  is an  $s \times s$ -matrices with coefficients in  $\mathbb{Z}/p^r\mathbb{Z}$ . This last fact is a consequence of the fact that multiplication by  $p^{r-1}$  induces an  $I^t$ -equivariant isomorphism  $T/pT \rightarrow T_p$ . The calculation  $(\text{Id} + pM)^{p^{r-1}} = \text{Id}$  now shows that the image of  $I^t$  in  $\text{Aut}(T)$  is a  $p$ -group. On the other hand,  $I^t$  is a direct sum of pro- $l$ -groups, with  $l \neq p$ . The order of the image of  $I^t$  is thus prime to  $p$ . This image is thus trivial.  $\square$

**Lemma 2.4** (Boxall). *Let  $B$  be an abelian variety over a field  $F$  of characteristic 0. Let  $l > 2$  be a prime number and let  $L := F(A[l])$  be the extension of  $K$  generated by the  $l$ -torsion points of  $A$ . Let  $P \in \text{Tor}_l(B(\bar{L}))$  and suppose that  $P \notin B(L)$ . Then there exists  $\sigma \in \text{Gal}(\bar{L}|L)$  such that  $\sigma(P) - P \in B[p](\bar{L}) \setminus \{0\}$ .*

**Proof.** See [4, Lemme 1] or [8, Prop. 3].  $\square$

**Lemma 2.5.** *Let  $P \oplus Q$  and  $P' \oplus Q'$  be two divisors of degree 2 on  $C_{\bar{L}}$ . Suppose that  $P \oplus Q$  and  $P' \oplus Q'$  are linearly equivalent.*

*If  $C$  is not hyperelliptic, then the two divisors coincide.*

*If  $C$  is hyperelliptic, then either the two divisors coincide or we have  $Q = \iota(P)$  and  $Q' = \iota(P')$ .*

Here  $\iota : C \rightarrow C$  is the uniquely defined hyperelliptic involution.

**Proof.** See [7, IV, Prop. 5.3].  $\square$

**Lemma 2.6.** *Let  $x \in A(\bar{L})$  and suppose that  $x \neq 0$ . The inequality*

$$\#(C(\bar{L}) \cap (C(\bar{L}) + x)) \leq 2 \tag{1}$$

*is then verified. If  $C$  is not hyperelliptic, we even have*

$$\#(C(\bar{L}) \cap (C(\bar{L}) + x)) \leq 1. \tag{2}$$

This Lemma is a consequence of [1, Prop. 4]. For the convenience of the reader, we provide the following proof.

**Proof.** We shall write  $O$  for the  $K$ -rational point on  $C$ , which is used to embed  $C$  in  $A$ .

Let  $a_1, \dots, a_r \in C(\bar{L})$  be pairwise distinct points such that  $a_1 + x, a_2 + x, \dots, a_r + x \in C(\bar{L})$ .

Let  $b_i := a_i + x$  ( $i = 1, \dots, r$ ).

Suppose first that  $r > 1$  and that  $C$  is not hyperelliptic. We then have a linear equivalence

$$b_1 \oplus a_2 \sim b_2 \oplus a_1 \tag{3}$$

Hence either  $b_1 = b_2$  or  $b_1 = a_1$ , either of which are ruled out. So we conclude that if  $C$  is not hyperelliptic, then  $r \leq 1$ . This proves the inequality (2).

Now suppose that  $C$  is hyperelliptic and that  $r > 2$ . Let  $\iota : C \rightarrow C$  be the corresponding hyperelliptic involution. On top of (3), we then have the further linear equivalence

$$b_2 \oplus a_3 \sim a_2 \oplus b_3$$

Lemma 2.5 now implies that  $a_2 = \iota(b_1)$  and  $a_2 = \iota(b_3)$ . Hence  $b_1 = b_3$ , which is impossible. Thus we conclude that  $r \leq 2$ , if  $C$  is hyperelliptic. This proves the first inequality (1).  $\square$

We can now start with the proof of Theorem 1.1.

The monodromy theorem 2.1 says that for any  $x \in \text{Tor}^p(A(\bar{L}))$  and any  $\sigma \in I$  the equation  $(\sigma - \text{Id})^2(x) = 0$  is satisfied (remember that  $\text{Tor}^p(\cdot) = \text{Tor}(\cdot)$  if  $p = 0$ ). On the other hand, Lemma 2.2 says that  $\text{Tor}^p(A(\bar{L})) \subseteq \text{Tor}^p(A(L^t))$  and Lemma 2.3 implies that  $\sigma(x) = x$  for any  $x \in \text{Tor}_p(A(L^t))$  and any  $\sigma \in I^t$ . Hence the equation

$$(\sigma - \text{Id})^2(x) = 0 \tag{4}$$

is verified for any  $x \in \text{Tor}(A(L^t))$  and any  $\sigma \in I^t$ . Let  $x \in \text{Tor}(A(L^t)) \cap C(\bar{L})$  and  $\sigma \in I^t$ . The equation (4) implies the linear equivalence

$$\sigma^2(x) \oplus x \sim 2\sigma(x) \tag{5}$$

of divisors of degree 2 on  $C_{\bar{L}}$ .

First suppose that  $C$  is not hyperelliptic; then any two linearly equivalent divisors of degree 2 on  $C_{\bar{L}}$  coincide (see Lemma 2.5) and the relation (5) thus implies that  $x = \sigma(x)$ .<sup>1</sup>

*Summing up, there is an inclusion*

$$\text{Tor}(A(L^t)) \cap C(\bar{L}) \subseteq A(L)$$

*if  $C$  is not hyperelliptic.*

Now suppose that  $C$  is hyperelliptic and let  $\iota : C \rightarrow C$  be the uniquely defined hyperelliptic involution. Lemma 2.5 then implies that either  $x = \sigma(x)$  as above or that the equations  $\iota(\sigma(x)) = \sigma(x)$  and  $\iota(x) = \sigma^2(x)$  hold. Suppose the latter. Since  $\iota$  is defined over  $L$ , the equation  $\iota(\sigma(x)) = \sigma(x)$  implies that  $\iota(x) = x$ . This together with the equation  $\iota(x) = \sigma^2(x)$  implies that  $\sigma^2(x) = x$ .

Now notice that since the group  $I^t$  is abelian, the set  $(I^t)^2$  of squares of elements is a normal subgroup of  $I^t$ . Let  $J'$  be the Galois extension of  $L$  defined by  $(I^t)^2$ . Since  $I^t \simeq \bigoplus_{l \neq p, l \text{ prime}} \mathbb{Z}_l$ , we see that  $[J' : L] = 2$ . In the last paragraph, we showed that  $\text{Tor}(A(L^t)) \cap C(\bar{L})$  is fixed by  $(I^t)^2$ . In other words, we have shown that  $\text{Tor}(A(L^t)) \cap C(\bar{L}) \subseteq A(J')$  where  $J'$  is the unique moderately ramified quadratic extension of  $L$ . In the notation of the introduction,  $J' = L'$ .

*Summing up, we see that*

$$\text{Tor}(A(L^t)) \cap C(\bar{L}) \subseteq A(L')$$

*if  $C$  is hyperelliptic.*

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<sup>1</sup>This calculation is partly the motivation for Ribet's definition of an "almost rational point" (see [2, Lemma 2.7]) and is the starting point of Tamagawa's article [10] (see Prop. 0.2 in that reference).

If  $p = 0$ , then  $L^t = \bar{K}$  so this completes the proof of Theorem 1.1 in that case.

Now let  $x \in \text{Tor}(A(\bar{L})) \cap C(\bar{L}) \setminus C(L^t)$ . Let  $x = x^p + x_p$  be the decomposition of  $x$  into its components of prime-to- $p$  torsion and  $p$ -primary torsion, respectively. Lemma 2.3 implies that  $x_p \in A(\bar{L}) \setminus A(L^t)$ . Also, using Boxall's lemma 2.4 and the fact that  $L$  contains the coordinates of the  $p$ -torsion points, we see that there exists  $\sigma_x \in I^w$  such that  $\sigma_x(x) - x = \sigma_x(x_p) - x_p \in A[p](L) \setminus \{0\}$ . Hence

$$\sigma_x(x) \in \bigcup_{\tau \in A[p](L) \setminus \{0\}} C(\bar{L}) \cap (C(\bar{L}) + \tau). \quad (6)$$

Lemma 2.6 now implies that

$$\#(C(\bar{L}) \cap (C(\bar{L}) + \tau)) \leq 2 \quad (7)$$

for all  $\tau \in A[p](L) \setminus \{0\}$ . Notice that if  $\sigma_x(x) \in C(\bar{L}) \cap (C(\bar{L}) + \tau_0)$  for a particular  $\tau_0 \in A[p](L) \setminus \{0\}$ , then we have

$$\{\sigma_x(x), \sigma_x^2(x), \dots, \sigma_x^p(x)\} \subseteq C(\bar{L}) \cap (C(\bar{L}) + \tau_0)$$

(remember that by construction  $\tau_0$  is fixed by  $I^w$ ). On the other hand

$$\{\sigma_x(x), \sigma_x^2(x), \dots, \sigma_x^p(x)\} = \{x + \tau_0, x + 2\tau_0, \dots, x + p\tau_0 = x\}.$$

Since  $p > 2$  and  $\tau_0$  has exact order  $p$  in  $A(\bar{K})$ , this leads to a contradiction. Thus we have

$$\text{Tor}(A(\bar{L})) \cap C(\bar{L}) \subseteq A(L^t).$$

Now remember that we have shown above (see the italicized sentences) that  $\text{Tor}(A(L^t)) \cap C(\bar{L}) \subseteq A(L)$  if  $C$  is not hyperelliptic and that  $\text{Tor}(A(L^t)) \cap C(\bar{L}) \subseteq A(L')$  if  $C$  is hyperelliptic. This concludes the proof of (a) and (b).

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